Section 11.10 - Taylor/Malaurin Series (#48)

Taylor Series $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$ Maclaurin Series $a=0$ $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$	

(I) Derivation of Taylor Series

$1 \quad 0! \quad 1! \quad 2! \quad 3!$

$f^{(0)} \quad f^{(1)} = f' \, , \, f^{(2)} = f''$

① Assume f(x) can be given by a power series centered at a where $|x-a| < R$

$\therefore f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots c_n(x-a)^n$

$\Rightarrow \boxed{f(a) = c_0}$

② $f'(a) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 \ldots n(c_n)(x-a)^{n-1}$

$\boxed{f'(a) = c_1}$

③ $f''(x) = 2c_2 + 6(x-a) \ldots (n)(n-1)(x-a)^{n-1}$

④ $f'''(a) = 6c_2 \Rightarrow \boxed{c_2 = \frac{f'''(a)}{6}}$

⑤ $f^n = n! \, C_n + \dots$

$f^n(a) = n! \, C_n \Rightarrow \boxed{C_n = \frac{f^n(a)}{n}}$

⑥ now plug coefficients back into $f(x)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''}{3!}(x-a)^3 \dots \frac{f^n}{n!}(x-a)^n \dots$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!} \Leftarrow \text{Taylor Series}$$

(II) Maclaurin Series Taylor w/ $a=0$
i.e. Power Series Centered at 0

(III) Example: e^x centered at 0

$f(x) = e^x \Rightarrow f(0) = 1$
$f'(x) = e^x \Rightarrow f'(0) = 1$
$f''(x) = e^x \Rightarrow f''(0) = 1$
$\vdots \qquad \vdots$
$f^{(n)}(x) = e^x \Rightarrow f^n(0) = 1$

$$\Rightarrow e^x = 1 + \frac{(1)(x)}{1!} + \frac{(1)(x)^2}{2!} + \frac{(1)(x)^3}{3!} + \dots \frac{(1)x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Radius of Convergence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \qquad x \in \mathbb{R}$$
$R = \infty$

Section 11.10 - Day (2) #9

Example

sin(x) centered at 2

$$\Rightarrow F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) = \sin(x)$ $\qquad$ $f(a) = \sin(a)$

$f'(x) = \cos(x)$ $\qquad$ $f'(a) = \cos(a)$

$f''(x) = -\sin(x)$ $\qquad$ $f''(a) = -\sin(a)$

$f'''(x) = -\cos(x)$ $\qquad$ $f'''(a) = -\cos(a)$

$$f(x) = \sin(2) + \cos(2)(x-2) - \frac{\sin(2)}{2!}(x-2)^2 - \frac{\cos(2)}{3!}(x-2)^3 + \frac{\sin(2)}{4!}(x-2)^4 \cdots$$

This polynomial strives to match sin(x) centered at (a)

Matlab $\Rightarrow$ Demo (go 2)

Maple $\Rightarrow$ Taylor(sin(x), x=2, 50)

II Why Do we care?

Now can integrate $\int e^{x^2} dx$

$\Rightarrow$ convert to Taylor Polynomial

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\Rightarrow \int e^{x^2} dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C}$$

<u>Board</u>

① Integrate e^{x^3}, e^{-x^2}, $\sin(x^2)$, $\cos(x^3)$, $\tan^{-1}(x^4)$, $\frac{1}{1-x^4}$

(IV) Example: e^x centered at a

$$f(a) = e^a$$
$$f'(a) = e^a$$
$$\vdots$$
$$f^{(n)}(a) = e^a$$

$$e^x = e^a + e^a(x-a) + e^a\frac{(x-a)^2}{2!} + \dots e^a\frac{(x-a)^n}{n!}$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!} = \underline{\underline{e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}}}$$

(V) Demo: e^x centered at 2

(VI) __Board__ __Work__ - centered at 0

e^{-x}, e^{2x}, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$,

$\sin(3x)$