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9.1. Modeling with Differential Eqns

$$\frac{dP}{dt} = kP$$

t = time (the independent variable)

P = the number of individuals in the population (the dependent variable)

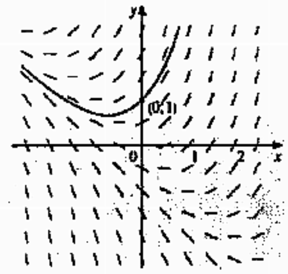
$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

A Model for the Motion of a Spring

restoring force $= -kx$

$$m\frac{d^2x}{dt^2} = -kx \qquad \text{OR} \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$



9.2 Direction Fields and Euler's Method

The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

Euler's Method Eqn

$$y_1 = y_0 + hF(x_0, y_0)$$

$$y_2 = y_1 + hF(x_1, y_1)$$

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

EXAMPLE 3 Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

SOLUTION We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

9.3 Separable Differential Equations

A type of differential equation that *can* be solved explicitly.

A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation:

$$\int h(y)\,dy = \int g(x)\,dx$$

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so $$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

and $$h(y)\frac{dy}{dx} = g(x)$$

EXAMPLE

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SOLUTION

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2\,dy = x^2\,dx$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$.

Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

3.8 Exponential Growth and Decay

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt.

THEOREM The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{1}{P}\frac{dP}{dt} = k$$

Radioactive Decay

$$m(t) = m_0 e^{kt}$$

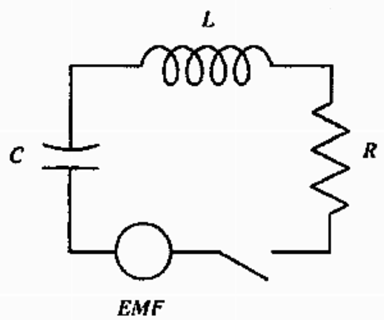
Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_s) \quad \text{->} \quad \frac{dy}{dt} = ky$$

Continuously Compounded Interest

$$\frac{dA}{dt} = rA_0e^{rt} = rA(t)$$

HO Electric Circuits: DC



Circuit Devices			
device name	symbol	units	voltage drop
capacitor	C	Farads (F)	$E_C = \frac{1}{C} \cdot Q$
resistor	R	Ohms (Ω)	$E_R = R \cdot I$
inductor	L	Henries (H)	$E_L = L \cdot \frac{dI}{dt}$

$$I(t) = \frac{dQ(t)}{dt}$$

Kirchhoff's Law: The sum of the votage drops around the closed circuit is equal to the voltage supplied:

$$E_L + E_R + E_C = E$$

[a] The device labeled EMF, which stands for *electromotive force*, can be a battery or a generator. The EMF supplies current to the circuit.
[b] The device labeled C is a *capacitor* which stores charge.
[c] The device labeled R is a *resistor* which opposes the flow of current.
[d] The device labeled L is an *inductor* which opposes change in the flow of the current.

Examples:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q - E$$

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$L\frac{dI}{dt} + RI = E$$

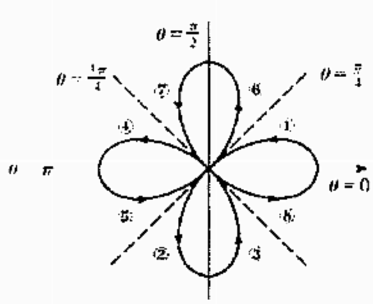
10.3 Polar Cordinates

Polar to Cartesian —> $x = r\cos\theta$ $\quad y = r\sin\theta$

Cartesian to Polar —> $r^2 = x^2 + y^2$ $\quad \tan\theta = \frac{y}{x}$

EXAMPLE Sketch the curve $r = \cos 2\theta$.

PLUG INTO CALCULATOR AND CHANGE GRAPHING FROM FUNCTION TO POLAR.



10.4 Areas in Polar Coordinates

$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta \quad \textbf{OR} \quad A = \int_a^b \tfrac{1}{2}r^2\,d\theta$$

EXAMPLE Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\,d\theta = \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$